A $5r$-covering theorem

For a ball $B=B(x,r)\subset\mathbb{R}^d$ (ball centered at x with radius r) we write $5B:=B(x,5r)$. Let $\mathcal{C}$ be a collection of balls contained in a bounded subset of $\mathbb{R}^d$. Then $\exists$ a countable (this means finite or countably infinite) subcollection $\{B_i\}\subset\mathcal{C}$ s.t. $\bigcup_{B\in\mathcal{C}} B\subset\bigcup_i 5\cdot B_i$.

__Proof__ We define $\{B_i\}$ inductively. Let $d_0:=\sup\{|B|: B\in\mathcal{C}\}$. We choose $B_1\in\mathcal{C}$ s.t. $|B_1|\ge d_0/2$. If we have already defined $B_1,\dots,B_m$ then we define

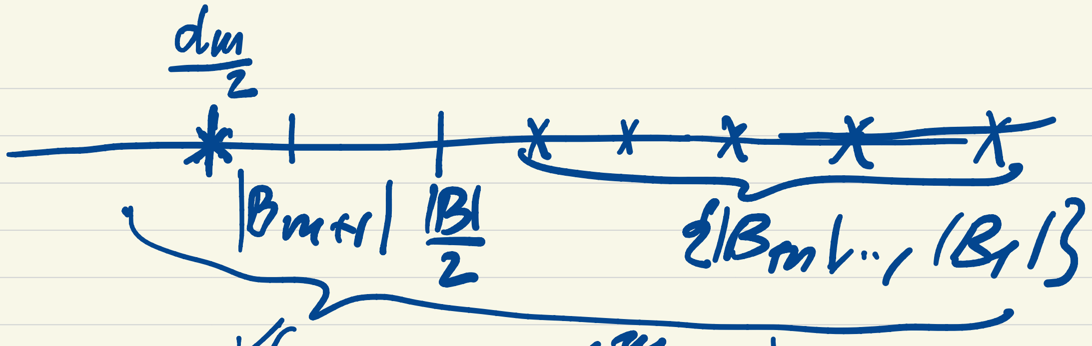
$$d_m:=\sup\{|B|: B\in\mathcal{C},\ B\cap\bigcup_{i=1}^m B_i=\emptyset\}.$$

If $d_m=0$ then we are ready. Otherwise we choose B_{m+1} s.t.

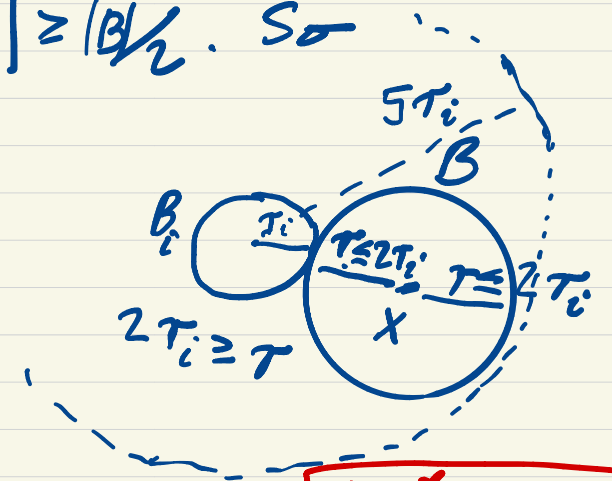
$$B_{m+1}\in\mathcal{C}\ \&\ |B_{m+1}|\ge d_m/2.$$

__Claim__ For every $B\in\mathcal{C}$ either $\exists i$ s.t. $B=B_i$ or $\exists i$ s.t. $B\cap B_i\neq\emptyset\ \&\ |B_i|\ge\frac{|B|}{2}$.

__Proof of the Claim__ Let $m+1:=\min\{k: 2|B_k|<|B|\}$. (This makes sense since $\{B_i\}$ disjoint so $\sum_i|B_i|^d<\infty$. Hence $\lim_{k\to\infty}|B_k|=0$) Then $\exists\, i\in\{1,\dots,m\}$ s.t. $B\cap B_i\neq\emptyset$. See the next figure:



$\Downarrow$ $d_m<|B|$ hence $B\cap(\bigcup_{i=1}^m B_i)\neq\emptyset$. Now we prove that $B\subset\bigcup_{i=1}^m 5B_i$. Namely, for $\cdot\in\{1,\dots,m\}$ $|B_i|\ge|B|/2$. So $B_i\cap B\neq\emptyset\Rightarrow 5B_i\supset B$.



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Actually the assertion of the theorem holds in a boundedly compact metric space where the center of a ball need not be unique. There

$5B:=\bigcup\{B': B' \text{ is a ball with } B'\cap B=\emptyset\ \&\ |B'|\le 2|B|\}$.

Then $|5B|\le 5|B|$.

Metric space is boundedly compact if all bounded closed sets are compact.

For the proof see Mattila's first book p. 24.

Vitali Covering Theorem for Lebesgue measure

Theorem (Vitali for Leb. measure) Let $A\subset\mathbb{R}^d$ and let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^d$ s.t. $\forall x\in A$ $\inf\{|B| : x\in B,\ B\in\mathcal{B}\}=0$.
Then $\exists$ disjoint balls $B_i\in\mathcal{B}$ s.t. $\mathcal{L}^d(A\setminus\bigcup_i B_i)=0$ (d-dimensional Leb. meas)

Moreover, for every $\varepsilon>0$ we can choose the balls $\{B_i\}_i$ s.t.

$$\sum_i \mathcal{L}^d(B_i)\le \mathcal{L}^d(A)+\varepsilon.$$

Proof W.L.G. we may assume that the set A is bounded. Choose an open set $U\supset A$ s.t. $\mathcal{L}^d(U)\le(1+7^{-d})\mathcal{L}^d(A)$. Consider those balls from $\mathcal{B}$ which are contained in U and apply the $5r$-covering theorem for this collection of balls. This results $\{B_i\}$ s.t. $\{B_i\}$ are disjoint, $B_i\subset U$, $\forall i$, $A\subset\bigcup_i 5B_i$.

So $5^{-d}\mathcal{L}^d(A)\le 5^{-d}\sum_i\mathcal{L}^d(5B_i)=\sum_i\mathcal{L}^d(B_i)$

Hence $\exists k_1$ s.t.
$6^{-d}\mathcal{L}^d(A)\le\sum_{i=1}^{k_1}\mathcal{L}^d(B_i)$. Let $A_1:=A\setminus\bigcup_{i=1}^{k_1}B_i$

Let $u:=1+7^{-d}-6^{-d}<1$ So, $\mathcal{L}^d(A_1)\le u\cdot\mathcal{L}^d(A)$

Namely,
$$\mathcal{L}^d(A_1)\le\mathcal{L}^d\Big(U\setminus\bigcup_{i=1}^{k_1}B_i\Big)=\mathcal{L}^d(U)-\sum_{i=1}^{k_1}\mathcal{L}^d(B_i)\le\underbrace{(1+7^{-d}-6^{-d})}_{u}\mathcal{L}^d(A).$$

Using that $A_1\subset\bigcup_{i=1}^{k_1}B_i$, $\exists U_1$ open set:
$A_1\subset U_1$ & $\mathcal{L}^d(U_1)\le(1+7^{-d})\mathcal{L}^d(A_1)$ & $U_1\cap\bigcup_{i=1}^{k_1}B_i=\emptyset$.
As above we select $\{B_i\}_{i=1}^{k_2}$ disjoint balls, $B_i\in\mathcal{B}$, $B_i\subset U_i$ for all $i=k_1+1,\dots,k_2$ s.t.
$A_2:=A_1\setminus\bigcup_{i=k_1+1}^{k_2}B_i=A\setminus\bigcup_{i=1}^{k_2}B_i$. Similarly as above

$\mathcal{L}^d(A_2)\le u\,\mathcal{L}^d(A_1)\le u^2\mathcal{L}^d(A)$. Clearly $\{B_i\}_{i=1}^{k_2}$ disjoint.

Repeating this m times we get
$\{B_i\}_{i=1}^{k_m}$; $B_i\in\mathcal{B}$ disjoint balls; $\mathcal{L}^d(A\setminus\bigcup_1^{k_m}B_i)\le u^m\mathcal{L}^d(A)$.

Using that $u<1$ the result follows for $\{B_i\}_{i=1}^{\infty}$.

Remark 1 Assume that μ is a Radon measure for which $\exists$ $1<\tau<\infty$ s.t.

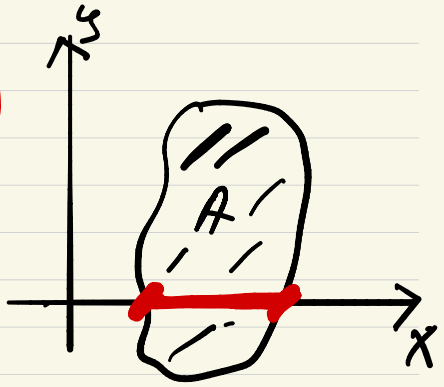
$$\overline{\lim_{r\downarrow 0}}\left\{\frac{\mu(B(y,\tau\cdot r))}{\mu(B(y,r))} : x\in B(y,r)\right\}<\infty,$$

holds for μ-a.e. $x\in\mathbb{R}^d$.
Then we can substitute $\mathcal{L}^d$ with μ.

Remark 2 The theorem does NOT hold for all Radon measures.

Example Let μ be the Radon measure on $\mathbb{R}^2$ defined by:

$$\mu(A) := \mathcal{L}^1\left(\{x \in \mathbb{R} : (x,0) \in A\}\right)$$



Let

$\mathcal{B} = \{ \overline{B((x,y),y)} : x \in \mathbb{R},\ 0 < y < \infty \}$

covers $A = \{(x,0) : x \in [0,1]\}$.

But for any countable sub-collection of $\mathcal{B}$ we have $\mu(A \cap \bigcup_{i=1}^{\infty} B_i) = 0$.